Calculating beam fill factors

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August 6, 2008

When a radio source has an angular size that is comparable to the beam size of a radio telescope, some of its radiation arrives offset from the main axis of the beam. As the gain of the beam decreases with this offset, this will lead to an underestimation of the brightness of the radio source. The beam fill factor is the correction that should be applied to get the actual source brightness. This paper discusses a possible mathematical determination of this factor, and compares it to two known approximations.

In oder to determine the beam fill factor, we will have to introduce a number of simplifications. Firstly, we are looking at a circular object placed in a circular beam. We will also assume those two are aligned perfectly, and the beam has a Gaussian beam profile (this tends to hold true for at least horn and dish antennas, and probably others). The last assumption is that the source is radiating evenly, that is it has a constant surface brightness.

We define a function f(r) which denotes how the sensitivity of our beam depends on the angular offset of the main beam. Let's first see what happens if f(r) is always equal to 1, that is, we are using a truly omnidirectional antenna. In that case, we see all of the energy from our circular source:

$$S_v = \int_0^\rho \int_0^{2\pi} rBf(r)d\phi dr$$

Here S_v is the received signal power per unit bandwidth in the antenna, ρ is the half angle the source subtends and B is the (constant) surface brightness. This essentially calculates the area of the source, with a weighting factor that depends on the beam offset. For an evenly radiating source this is directly proportional to the received energy.

As everything is circularly symmetric, this immediately reduces to:

$$S_v = 2\pi B \int_0^\rho rf(r)dr$$

For the omnidirectional case, or at least when the beam width is much larger than the source, $f(r) = 1$ and the integral simply becomes $B\pi\rho^2$ - this is equivalent to having a beam correction factor of 1.

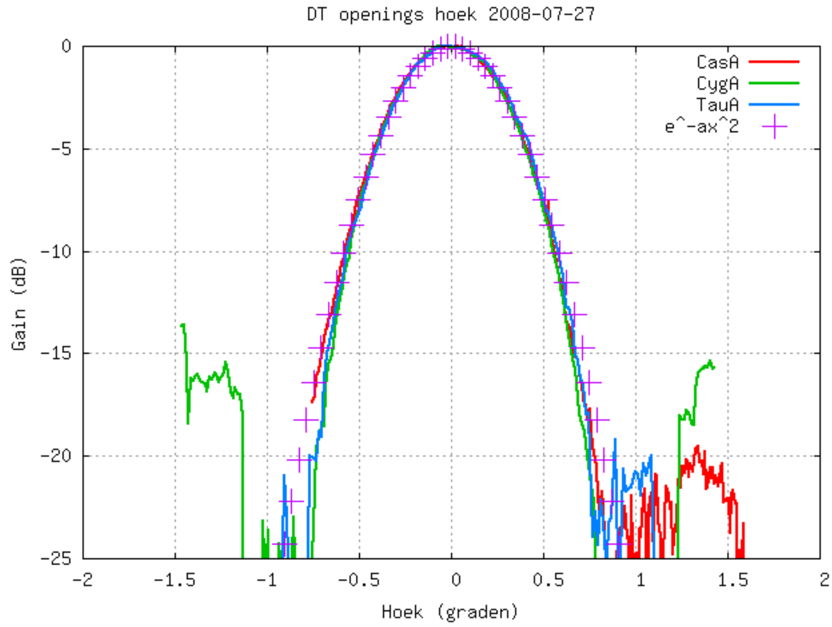
For horn antennas and dishes, the main beam is often Gaussian shaped. A normalized (unity gain) antenna pattern can then be written as

$$f(r) = e^{-ar^2}$$

Here r is the angular offset from the main beam. The value a determines how wide or narrow the beam is, and can be calculated for a given beam width (FWHM). Half this beam width is $r_{0.5}$ in the formulas below.

$$e^{-ar_{0.5}^2} = \frac{1}{2} \Leftrightarrow -ar_{0.5}^2 = \ln(\frac{1}{2}) \Leftrightarrow a = -\frac{\ln(\frac{1}{2})}{r_{0.5}^2}$$

For our 25m dish, the 3dB points are 0.64 degrees apart. So $f(-0.32) = f(0.32) = \frac{1}{2}$ and then $a = 0.693/0.1024 = 6.77$. This matches our actual main beam pattern as measured very well. The graph below shows drift-scans of three different sources, converted to angular offset. The graph of $f(r) = e^{-ar^2}$ is superimposed with $+$ marks.



Next we define a value β which is Ws/Wa. And from the geometry it immediately follows that $\beta = Ws/Wa = \rho/r_{0.5}$. Integrating the beam function over the surface of the source now becomes:

$$S_v = 2\pi B \int_0^\rho r f(r) dr = 2\pi B \int_0^\rho r e^{-ar^2} dr = 2\pi B \int_0^\rho r e^{\frac{\ln(\frac{1}{2})\beta^2}{\rho^2} r^2} dr = \frac{\pi B \rho^2}{\beta^2 \ln(\frac{1}{2})} \left(e^{\ln(\frac{1}{2})\beta^2} - 1 \right)$$

The beam fill factor L is the ratio of the value for $\beta = 0$ to the amount of energy that is received for the actual value of β:

$$L = B\pi\rho^2 / S_v = B\pi\rho^2 / \left(\frac{B\pi\rho^2}{\beta^2 \ln(\frac{1}{2})} \left(e^{\ln(\frac{1}{2})\beta^2} - 1 \right) \right)$$

$$\boxed{L = \frac{\beta^2 \ln(\frac{1}{2})}{\left(e^{\beta^2 \ln(\frac{1}{2})} - 1 \right)}}$$

Although not immediately obvious, this function does have the property that $L = 1$ for $\beta = 0$, because $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$.

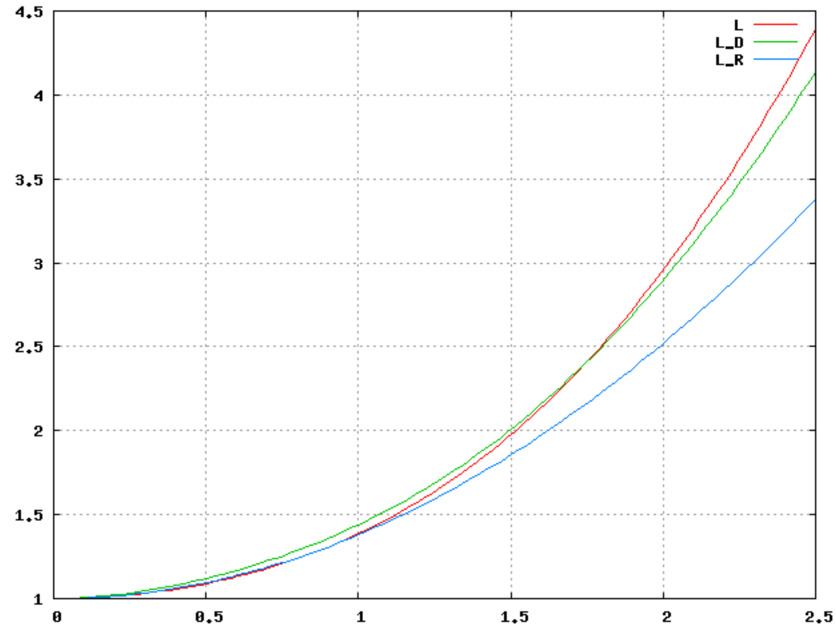
We can now compare the behavior of this function to two other approximations of the beam fill factor. First, there is the one used by Doug McArthur VK3UM, which is:

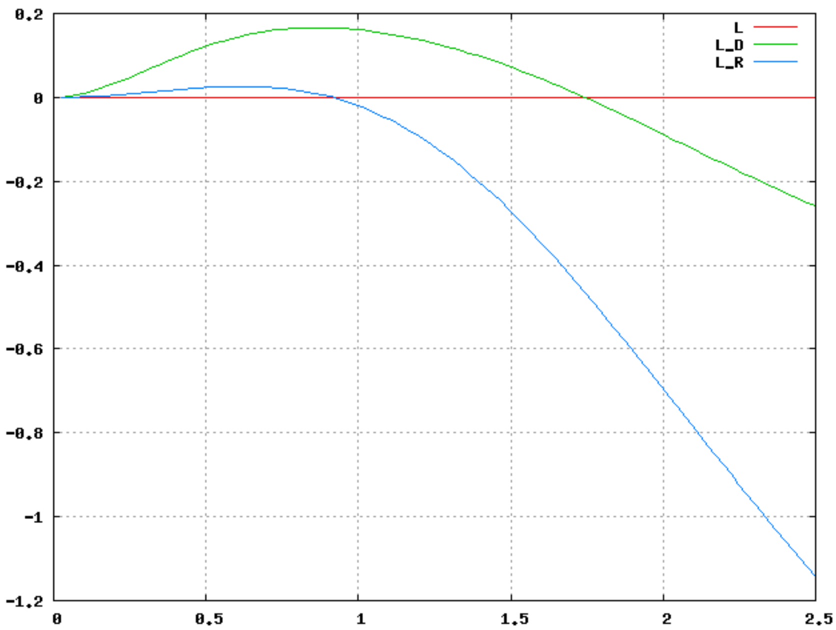
$$L_D = \sqrt{1 + 1.19\beta^2 - 0.57\beta^3 + 0.45\beta^4}$$

Then there is the one used by Richard Flagg AH6NM, which is:

$$L_R = 1 + 0.38\beta^2$$

The image below shows all three graphs plotted, once on a linear scale and then on a relative logarithmic scale.





For small values of β the simpler formula for L_R conforms better to the derived formula. For larger values, L_D still stays within 0.2dB whereas the other formula veers off more and more. Of course, this still leaves open the question which of the three formulas gives the best approximation for L: for now, we have no information how the two approximations were derived and whether they include any factors that are not part of the mathematical derivation above for L.